ℓ-adic Representations

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Our goal today is to understand ℓ-adic Galois representations a bit better, mostly by relating them to representations appearing in geometry. First we'll deal with Galois groups of finite fields, where we can say quite a lot, and then we'll deal with Galois groups of local fields, which is harder. We follow Section 1 of Fontaine and Ouyang's book *Theory of p-adic Galois representations*.

1 Examples

Let K be a field, and $G_K = \mathrm{Gal}(K^{\mathrm{sep}}/K)$. An *$\ell$-adic representation* of G_K is simply a continuous representation $G_K \to \mathrm{GL}_n \mathbb{Q}_\ell$. These representations are much richer than complex Galois representations, because the topologies are more compatible. For example, the image of any continuous representation with complex coefficients must be finite, because G_K is profinite; but this is not at all true for $\mathrm{GL}_n \mathbb{Q}_\ell$, which has many more profinite subgroups.

A wealth of examples of ℓ-adic representations of G_K is given by Tate modules of commutative algebraic groups.

Example. First we consider the Tate module of the multiplicative group $\mathbb{G}_m$. For a prime ℓ different from the characteristic of K, the group of ℓ^n roots of unity in K^{sep} is $\mu_{\ell^n}(K) = \mathbb{Z}/\ell^n$. The ℓ-powering maps $\mu_{\ell^{n+1}}(K^{\mathrm{sep}}) \to \mu_{\ell^n}(K^{\mathrm{sep}})$ form an inverse system, whose limit is the Tate module $T_\ell \mathbb{G}_m = \varprojlim \mu_{\ell^n}(K^{\mathrm{sep}})$. This is a free $\mathbb{Z}_\ell$-moduel of rank 1, and it carries a representation of G_K induced by the actions on $\mu_{\ell^n}(K^{\mathrm{sep}})$. We also define $V_\ell \mathbb{G}_m = \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} T_\ell \mathbb{G}_m$, a 1-dimensional $\mathbb{Q}_\ell$-vector space with G_K action. In fact this representation is the ℓ-adic cyclotomic character.

A common bit of nomenclature: we also write $T_\ell \mathbb{G}_m = \mathbb{Z}_\ell(1)$ and $V_\ell \mathbb{G}_m = \mathbb{Q}_\ell(1)$. If V is any ℓ-adic G_K representation, then for any $r \in \mathbb{Z}$ we define a *Tate twist* of V by $V(r) = V \otimes \mathbb{Q}_\ell(1)^{\otimes r}$ (where we understand negative r to mean the corresponding tensor power of the dual).

Example. We can also consider the Tate module of an elliptic curve E/K given by $T_\ell E = \varprojlim E[\ell^n]$, and $V_\ell E = \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} T_\ell E$. This is a 2-dimensional ℓ-adic G_K representation

Example. More generally, if A/K is an abelian variety, then we define the Tate module similarly as $T_\ell A = \varprojlim A[\ell^n]$, the inverse limit of ℓ^n-torsion subgroups and multiplication-by-ℓ maps, and $V_\ell A = \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} T_\ell A$. This is an ℓ-adic G_K-representation of dimension $2g$, where g is the dimension of A.

These examples are all special cases of the true source of ℓ-adic representations, which is ℓ-adic étale cohomology. Briefly, consider the category of sheaves on the étale site of a scheme; we can define cohomology functors in the usual way, as the right derived functors of the global sections functor. Unfortunately non-torsion coefficients give strange results. But torsion coefficients work fine, so we define non-torsion coefficients through an inverse limit instead. (For example, on complex varieties étale cohomology with finite coefficients is the same as singular cohomology with finite coefficients, but this is not the case for integer coefficients.)

Let Y be a smooth proper variety over K^{sep}. Then we have cohomology groups $H^i(Y_{\text{ét}}, \mathbb{Z}/\ell^n)$ which are finite abeian groups killed by ℓ^n. The reduction maps $\mathbb{Z}/\ell^{n+1} \to \mathbb{Z}/\ell^n$ induce an inverse system

$$H^i(Y_{\text{ét}}, \mathbb{Z}/\ell^{n+1}) \to H^i(Y_{\text{ét}}, \mathbb{Z}/\ell^n),$$

and we define the ℓ-adic étale cohomology of Y to be

$$H^i_{\text{ét}}(Y, \mathbb{Q}_\ell) = \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} \varprojlim H^i(Y_{\text{ét}}, \mathbb{Z}/\ell^n).$$

Furthermore, if X is a smooth proper variety over K (as opposed to K^{sep}), then $H^i_{\text{ét}}(X_{K^{\mathrm{sep}}}, \mathbb{Q}_\ell)$ carries a G_K action: G_K acts on $X_{K^{\mathrm{sep}}}$, which gives compatible actions on $H^i(X_{K^{\mathrm{sep}},\text{ét}}, \mathbb{Z}/\ell^n)$, and thus an action on $H^i_{\text{ét}}(X_{K^{\mathrm{sep}}}, \mathbb{Q}_\ell)$.

Example. The Tate module of an abelian variety A is a special case of this:

$$H^i_{\text{ét}}(A_{K^{\mathrm{sep}}}, \mathbb{Q}_\ell) \cong \textstyle\bigwedge^i (V_\ell A)^\vee.$$

Example. If C is a curve,

$$H^1_{\text{ét}}(C, \mathbb{Q}_\ell) \cong H^1(\operatorname{Jac} C, \mathbb{Q}_\ell) \cong (V_\ell \operatorname{Jac} C)^\vee.$$

Example. If $X = \mathbb{P}^n_K$, then

$$H^i_{\text{ét}}(X_{K^{\mathrm{sep}}}, \mathbb{Q}_\ell) = \begin{cases} 0 & \text{for } i \text{ odd or } i > 2n \\ \mathbb{Q}_\ell\left(-\frac{m}{2}\right) & \text{for } 0 \le i \le 2n,\ i \text{ even.} \end{cases}$$

2 Finite Fields

Now let $K = \mathbb{F}_q$ be a finite field of characteristic p. We have $G_K = \widehat{\mathbb{Z}}$, topologically generated by a geometric Frobenius element $\tau : x \mapsto x^{1/p}$ (inverse to the usual arithmetic Frobenius $x \mapsto x^p$). Since our representations are continuous, they are determined by the image of τ. That is, for any $u \in \mathrm{GL}_n \mathbb{Q}_\ell$, the assignment $\tau \mapsto u$ extends to a representation $\rho : G_K \to \mathrm{GL}_n \mathbb{Q}_\ell$ in at most one way. Indeed, if $a \in \widehat{\mathbb{Z}}$ and $a_n \in \mathbb{Z}$ a sequence converging to t, then $\rho(a)$ is the (topological) limit $\lim u^{a_n}$, if this limit exists.

Proposition. *This limit exists if and only if the eigenvalues of u are ℓ-adic units, that is, if $P_u(t) = \det(u - t \cdot \mathrm{id}) \in \mathbb{Z}_\ell[t]$ and the constant term is a unit in $\mathbb{Z}_\ell$.*

Definition. The *characteristic polynomial* of an ℓ-adic G_K representation ρ is $\det(\mathrm{id} - t \cdot \rho(\tau))$, the characteristic polynomial of $\rho(\tau)$.

A representation ρ is semi-simple precisely when $\rho(\tau)$ is semi-simple, and so the characteristic polynomial of ρ determines it up to semi-simplification.

Now we return to geometry. Let X be a smooth proper geometrically connected variety over K. We define a ζ-function for X by

$$\zeta_X(t) = \exp\left(\sum_{n \ge 1} \frac{\#X(\mathbb{F}_{q^n})}{n} t^n\right) = \prod_{\substack{x \in X \\ \text{closed}}} \frac{1}{1 - t^{\deg x}}$$

(where $\deg x$ is the degree of the residue field at x over K).

Many of the properties of the étale cohomology of X can be understood as properties of its ζ-function. The latter are the content of the Weil conjectures.

Theorem. *Let X, ζ_X as above, and let d be the dimension of X.*

1. *There are polynomials $P_0, \ldots, P_{2d} \in Z[t]$ such that*

$$\zeta_X(t) = \frac{P_1(t) \cdots P_{2d-1}(t)}{P_0(t)P_2(t) \cdots P_{2d}(t)}.$$

2. *There is a functional equation*

$$\zeta_X\left(\frac{1}{q^d t}\right) = \pm q^{d\beta} t^{2\beta} \zeta_X(t), \qquad \beta = \frac{1}{2}\sum_{m=0}^{2d}(-1)^m \deg P_m.$$

3. *Over $\overline{\mathbb{Z}}$ we have $P_m(t) = \prod(1 - \alpha_{m,j}t)$ where $|\alpha_{m,j}| = q^{m/2}$ under any embedding $\overline{\mathbb{Z}} \to \mathbb{C}$.*

The idea is that $P_m(t)$ is the characteristic polynomial of the G_K representation $H^m_{\text{ét}}(X_{K^{\text{sep}}}, \mathbb{Q}_\ell)$, so the statements of the Weil conjectures correspond to statements about étale cohomology and its G_K action.

1. A Grothendieck-Lefschetz-type trace formula

$$\#X(\mathbb{F}_{q^n}) = \sum_i (-1)^i \operatorname{tr}(\mathrm{Frob}_{q^n}, H^i_{\text{ét}}(X_{K^{\text{sep}}}, \mathbb{Q}_\ell)).$$

2. Poincaré duality.

3. Purity.

What do we mean by purity? A *Weil number* of weight $w \in \mathbb{Z}$ is an $\alpha \in \overline{\mathbb{Q}}$ such that $q^i\alpha \in \overline{\mathbb{Z}}$ for some $i \in \mathbb{N}$, and $|\alpha| = q^{w/2}$ under any embedding $\overline{\mathbb{Q}} \to \mathbb{C}$. A Weil number α is *effective* if $\alpha \in \overline{\mathbb{Z}}$.

An ℓ-adic G_K-representation is *pure of weight w* if all roots of the characteristic polynomial of τ (i.e. Frobenius eigenvalues) are Weil numbers of weight w. Such a thing is *effective* if the Frobenius eigenvalues are effective. Note that if V is pure of weight w then $V(r)$ is pure of weight $w - 2r$, since τ is multiplication by q^{-1} in the cyclotomic character. The purity part of the Weil conjectures is the statement that $H^m_{\text{ét}}(X_{K^{\text{sep}}}, \mathbb{Q}_\ell)$ is pure and effective of weight m.

Now we can state finite field versions of some big conjectures about Galois representations.

An ℓ-adic G_K representation V is *geometric* if it is semi-simple and decomposes $V = \bigoplus_{w \in \mathbb{Z}} V_w$ where V_w is pure of weight w and almost all $V_w = 0$. An ℓ-adic G_K representation *comes from geometry* if it is isomorphic to a subquotient of $H^i_{\text{ét}}(X_{K^{\text{sep}}}, \mathbb{Q}_\ell)$ for some X and i.

Theorem. *All geometric representations come from geometry.*

Conjecture. *All representations coming from geometry are geometric.*

In fact in the finite field case it's even known that representations coming from geometry are pure, so the only thing remaining is to show they are semi-simple.

3 Local Fields

Now let K be a local field with residue field k of characteristic p, ring of integers $\mathcal{O}_K$, absolute Galois group $G_K = \mathrm{Gal}(K^{\text{sep}}/K)$, inertial subgroup I_K, and wild inertia subgroup P_K. (In the notes we're following a "local field" means a complete discrete valuation field with residue field perfect of characteristic p; in particular, the residue field may not be finite.)

Definition. Let $\rho : G_K \to \mathrm{GL}_n \mathbb{Q}_\ell$ be an ℓ-adic representation.

- ρ is *unramified* or has *good reduction* if $\rho(I_K) = \{id\}$.
- ρ has *potentially good reduction* if $\rho(I_K)$ is finite, or equivalently, if there is a finite extension $K \subset K' \subset K^{\mathrm{sep}}$ so that $\rho|_{G_{K'}}$ has good reduction.
- ρ is *semi-stable* if $\rho(I_K)$ is unipotent, or equivalently, if the semi-simplification of ρ has good reduction.
- ρ is *potentially semi-stable* if there is a finite extension $K \subset K' \subset K^{\mathrm{sep}}$ so that $\rho|_{G_{K'}}$ is semi-stable, or equivalently, if the semi-simplification of ρ has potentially good reduction.

There are a couple ways of describing all potentially semi-stable G_K representations. The first is through geometry.

Theorem. *If $\mu_{\ell^\infty}(K(\zeta_\ell))$ is finite, then any ℓ-adic G_K representation is potentially semi-stable. In particular, this is true if the residue field k is finite, because $\mu_{\ell^\infty}(K) \cong \mu_{\ell^\infty}(k)$.*

Note in particular that the usual notion of local field has a finite residue field, so in this case the following theorems describe all ℓ-adic G_K representations.

Corollary (Grothendieck's ℓ-adic monodromy theorem)**.** *Let K be a local field. Then any ℓ-adic representation of G_K coming from geometry is potentially semi-stable.*

This is proven essentially by finding a model of X over a field K' satisfying the hypotheses of the Theorem, and showing that the action of G_K comes from the action of $G_{K'}$.

Theorem. *Assume the residue field k is algebraically closed. Then any potentially semi-stable ℓ-adic representation of G_K comes from geometry.*

First, assume the representation is semi-stable, do some reductions, and then show that it comes from an elliptic curve. Then, assume that the representation is potentially semi-stable, so that there's a field extension K' over which it's semi-stable. Then take the elliptic curve that does the job for K' and Weil-restrict to K to get an abelian variety that does the job for K.

4 Weil-Deligne Representations

Let K remain a local field, but now suppose the residue field k is finite (i.e. a local field in the usual sense). We describe another method for understanding potentially semi-stable G_K-representations. An essentially similar thing is possible for the more general notion of local field, but we restrict to this case for simplicity.

We have an exact sequence

$$\begin{gathered} 1 \to I_K \to G_K \to G_k \to 1 \\ g \mapsto \overline{g} \end{gathered}$$

and since k is finite it has Galois group $G_k = \widehat{\mathbb{Z}}$, topologically generated by the geometric Frobenius τ.

Definition. The *Weil group* of K to be

$$W_K = \{g \in G_K : \overline{g} = \tau^m \text{ for some } m \in \mathbb{Z}\},$$

the preimage in G_K of $\mathbb{Z} \subset \widehat{\mathbb{Z}} = G_k$. It sits in an exact sequence

$$1 \to I_K \to W_K \xrightarrow{a} \mathbb{Z} \to 1.$$

Note also that W_K is dense in G_K.

The *Weil-Deligne group* of K is

$$WD_K = W_K \ltimes \mathbb{G}_a$$

where the action is $wxw^{-1} = q^{-a(w)}x$ for $w \in W_K$, $x \in \mathbb{G}_a$.

A *Weil representation* of K over a field F is a finite dimensional representation $\rho : W_K \to \mathrm{GL}_n F$ whose kernel contains an open subgroup of I_K. A *Weil-Deligne representation* of K over F is a Weil representation ρ together with a nilpotent endomorphism N of F^n satisfying

$$N \circ \rho(w) = q^{a(w)}\rho(w) \circ N \qquad \text{for all } w \in W_K.$$

A morphism $(D, N) \to (D', N')$ of Weil-Deligne representations over F is an F-linear G_K-equivariant map $\eta : D \to D'$ such that $N' \circ \eta = \eta \circ N$. (Probably.)

Any ℓ-adic representation of G_K with potentially good reduction determines a Weil representation, simply by restriction to W_K. Furthermore the G_K representation is determined by the Weil representation because W_K is dense in G_K. (Note that not any G_K representation produces a Weil representation, because a Weil representation is required to kill some inertia.)

An ℓ-adic representation V of G_K which is potentially semi-stable induces a Weil-Deligne representation,

$$D = \varinjlim_{\substack{H \subset I_K \\ \text{open normal}}} (\mathbb{Q}_\ell[u] \otimes_{\mathbb{Q}_\ell} V)^H, \qquad N : b \otimes v \mapsto \frac{db}{du} \otimes v$$

where the action on u is something like $\mathbb{Q}_\ell(-1)$.

Theorem. *There is an equivalence of categories*

$$\{\text{potentially semi-stable } G_K \text{ representations}\} \longleftrightarrow \{\text{Weil-Deligne representations of } K\}.$$

Let E, F be two fields of characteristic zero, and D (resp. D') a Weil-Deligne representation over E (resp. F). Then D, D' are said to be *compatible* if for any field Ω and inclusions $E \to \Omega$, $F \to \Omega$, we have

$$D \otimes_E \Omega \cong D' \otimes_F \Omega$$

as Weil-Deligne representations over Ω.

Theorem. *Let A be an abelian variety over K and ℓ, ℓ' primes different from each other and from p. Then $V_\ell(A), V_{\ell'}(A)$ are compatible.*

Conjecture. *Let X be a smooth projective variety over K, with ℓ, ℓ' as above. Then $H^i_{\acute{e}t}(X_{K^{\mathrm{sep}}}, \mathbb{Q}_\ell)$ and $H^i_{\acute{e}t}(X_{K^{\mathrm{sep}}}, \mathbb{Q}_{\ell'})$ are compatible.*

Now we define "geometric" in the local field setting. Choose $\tau \in W_K$ a lift of geometric Frobenius from G_k, and let w be an integer.

Definition. A Weil representation over E is *pure of weight w* if all roots of the characteristic polynomial of τ are Weil numbers of weight w. (This is independent of the choice of τ.)

If V is a Weil representation over E and r any natural number, then

$$D = V \oplus V(-1) \oplus \cdots \oplus V(-r), \qquad N : D \to D(-1)$$
$$(v_0, v_{-1}, \ldots, v_{-r}) \mapsto (v_{-1}, \ldots, v_{-r}, 0)$$

is a Weil-Deligne representation.

A Weil-Deligne representation is *elementary and pure of weight* $w + r$ if it is isomorphic to such a D as above with V semi-simple and pure of weight w.

A Weil-Deligne representation is *geometric and pure of weight* m if it is a direct sum of elementary and pure of weight m representations.

Geometric and pure of weight m Weil-Deligne representations form an abelian category.

As in the case of finite fields, we expect that representations coming from geometry are geometric, and geometric representations come from geometry. However, this case is not as well understood as the finite field case.

Conjecture. *For $\ell \neq p$, the Weil-Deligne representation associated to $H^i_{ét}(X_{K^{\mathrm{sep}}}, \mathbb{Q}_\ell)$ is geometric and pure of weight $i - 2r$. Furthermore, representations of this form generate the whole category.*